Discretizing Other Boundary Value Types.

EX: Discretize $y'' = 5t$ $\quad y'(2)=0 \quad y(3)=0$

with step-size $h = 1/5$

t	y
$t_0 = 2$	$y_0 = ??$
$t_1 = 11/5$	y_1
$t_2 = 12/5$	y_2
$t_3 = 13/5$	y_3
$t_4 = 14/5$	y_4
$t_5 = 3$	$y_5 = 0$

unknown: $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$

y_0'' cannot be computed

$$\left\{\begin{aligned} \tfrac{1}{h^2}(y_2 - 2y_1 + \overset{??}{\cancel{y_0}}) &= y_1'' = 5t_1 = 11 \\ \tfrac{1}{h^2}(y_3 - 2y_2 + y_1) &= y_2'' = 5t_2 = 12 \\ \tfrac{1}{h^2}(y_4 - 2y_3 + y_2) &= y_3'' = 5t_3 = 13 \\ \tfrac{1}{h^2}(\underset{0}{\cancel{y_5}} - 2y_4 + y_3) &= y_4'' = 5t_4 = 14 \end{aligned}\right\}$$

y_5'' cannot be computed

Boundary values should determine y_0 & y_5

$y(3) = 0 \rightarrow y_5 = y(t_5) = y(3) = 0$

$y'(2) = 0 \rightarrow y_0' = y'(t_0) = y'(2) = 0$

$\hookrightarrow y_0' = \tfrac{1}{h}(y_1 - y_0)$

$0 = 5(y_1 - y_0)$

$\boxed{y_0 = y_1}$

Plug $y_0 = y_1$, $y_5 = 0$, and $1/h^2 = 25$ into equations

$$\left\{\begin{aligned} 25y_2 - 25y_1 \quad &= 11 \\ 25y_3 - 50y_2 + 25y_1 &= 12 \\ 25y_4 - 50y_3 + 25y_2 &= 13 \\ -50y_4 + 25y_3 &= 14 \end{aligned}\right\}$$

as matrix eqn:

$$\boxed{\begin{bmatrix} -25 & 25 & 0 & 0 \\ 25 & -50 & 25 & 0 \\ 0 & 25 & -50 & 25 \\ 0 & 0 & 25 & -50 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 11 \\ 12 \\ 13 \\ 14 \end{bmatrix}}$$

alt:

$$\underbrace{\frac{1}{(1/5)^2}\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}}_{\frac{d^2}{dt^2} \text{ free-fixed}} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 11 \\ 12 \\ 13 \\ 14 \end{bmatrix}$$

EX: Discretize $y'' + y' = \delta(t - \tfrac{1}{2})$ $\quad y(0)=0 \quad y'(1)=0$

with step-size $h = 1/4$.

t	y	$y_n'' + y_n' = \delta(t-t_n)$
$t_0 = 0$	$y_0 = 0$	y_0'' cannot be computed
$t_1 = 1/4$	y_1	$\frac{1}{h^2}(y_2 - 2y_1 + \cancel{y_0}^{\,0}) + \frac{1}{2h}(y_2 - \cancel{y_0}^{\,0}) = 0$
$t_2 = 1/2$	y_2	$\frac{1}{h^2}(y_3 - 2y_2 + y_1) + \frac{1}{2h}(y_3 - y_1) = 1/h$
$t_3 = 3/4$	y_3	$\frac{1}{h^2}(\cancel{y_4}^{\,??} - 2y_3 + y_2) + \frac{1}{2h}(\cancel{y_4}^{\,??} - y_2) = 0$
$t_4 = 1$	$y_4 = ??$	y_4'' cannot be computed.

unknown

$t_2 = 1/2$ is impulse point

Boundary values: $\quad y_0 = y(t_0) = y(0) = 0$

$y_4' = y'(t_4) = y'(1) = 0$

$\hookrightarrow y_4' = \frac{1}{h}(y_4 - y_3)$ $\quad$ Backward Difference.

$0 = 4(y_4 - y_3) \implies \boxed{y_4 = y_3}$

Plug in $y_0 = 0$, $y_4 = y_3$, $1/h^2 = 16$, $1/2h = 2$:

$$\begin{cases} (16+2)y_2 - 32y_1 = 0 \\ (16+2)y_3 - 32y_2 + (16-2)y_1 = 4 \\ (-16+2)y_3 + (16-2)y_2 = 0 \end{cases}$$

As matrix equation:

$$\boxed{\begin{bmatrix} -32 & 18 & 0 \\ 14 & -32 & 18 \\ 0 & 14 & -14 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}}$$

alt:

$$\left(\underbrace{\frac{1}{(1/4)^2} \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}}_{\frac{d^2}{dt^2}\text{ fixed-free}} + \underbrace{\frac{1}{2(1/4)} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}}_{\frac{d}{dt}\text{ fixed-free}} \right) \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$$

Fixed-Free 2^{nd} deriv. matrix

Fixed

$$\begin{bmatrix} -2 & 1 & 0 & & 0 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & 1 & -2 & 1 \\ & & 0 & 1 & -2 \end{bmatrix} 1 = \begin{bmatrix} -2 & 1 & 0 & & 0 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & 1 & -2 & 1 \\ & & 0 & 1 & -1 \end{bmatrix}$$

Free

EX $\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ $\quad \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

Free - Fixed 2nd deriv. matrix

Free

$$\begin{bmatrix} -2 & 1 & 0 & & & \\ 1 & -2 & 1 & & 0 & \\ & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & 0 & & 1 & -2 & 1 \\ & & & 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & & & \\ 1 & -2 & 1 & & 0 & \\ & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & 0 & & 1 & -2 & 1 \\ & & & 0 & 1 & -2 \end{bmatrix}$$

Fixed

$$\left(\underline{\underline{Ex}} \quad \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \quad \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \right)$$

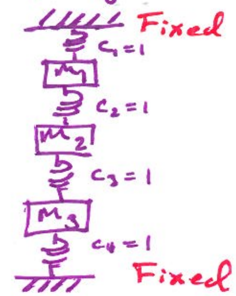
A similar thing happens with <u>first</u> deriv.

Fixed-Free
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

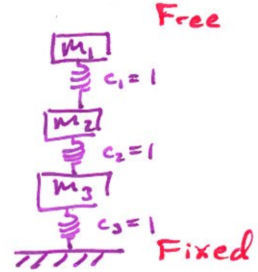
Free-Fixed
$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$-\frac{d^2}{dt^2}$ Matrices are all stiffness matrices for lines of masses and springs

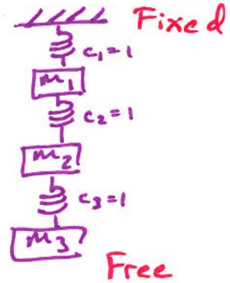
Fixed-Fixed
$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$



Free-Fixed
$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$



Fixed-Free
$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$



Free-Free
$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$